High-gradient acceleration of electrons in a plasma-loaded wiggler

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(Received 24 March 1995)

The interaction of an electron beam with a transverse electromagnetic field and a Langmuir wave in a plasma loaded wiggler is described by a system of self-consistent nonlinear equations. We demonstrate numerically and analytically that both beam-plasma and free-electron laser instabilities take place under suitable resonance conditions. As a consequence, the system is able to generate high-amplitude Langmuir waves with phase velocities larger than the speed of light, which give rise to high gradient and high energy acceleration of the electron beam.

PACS number(s): 41.85.−p, 52.40.Mj

I. INTRODUCTION

The idea of using a plasma in high-energy electron acceleration devices has led, in past years, to the plasma beat-wave acceleration scheme, in which two lasers beat together in a plasma and excite resonantly an intense Langmuir wave that accelerates the electrons of the beam [1–5]. On the other hand, the idea of using the high-energy concentration in a laser beam in the presence of a rippled static magnetic field (wiggler) that breaks the symmetry in the interaction with the electrons has led to the inverse free-electron laser concept [6].

The combination of these two schemes was recently analyzed by Bobin [7], who studied the possibility of accelerating the electrons of a relativistic beam in a wiggler loaded with a plasma [plasma–free-electron laser (FEL)] and in the presence of a high-frequency electromagnetic wave. One of the advantages presented by this system is that the Langmuir wave that accelerates the electrons of the beam is excited by a transverse wave injected into the plasma, this last being directly coupled to the electrostatic field by means of the wiggler magnetic field.

In this paper, we will show that large beam accelerations can be obtained when the frequency ω of the electromagnetic wave is of the order of the electron plasma frequency of the ambient plasma $\omega_p=(4\pi n_p e^2/m)^{1/2}$ (this was not the case considered by Bobin) and if the phase velocity v_L of the Langmuir wave and the intensity and wavelength of the wiggler magnetic field satisfy the condition $v_L=c\pm c a_{W0}/(2\sqrt{2})$, where c is the velocity of light in vacuum and a_{W0} is the characteristic parameter of the wiggler. We will show that this is the proper resonance condition of the plasma-FEL system, in the limit in which a_{W0} is smaller than one.

In Sec. II we introduce the equations used and the relevant plasma dispersion relation. In Sec. III the numerical results obtained are shown and their interpretation is discussed. Possible applications and conclusions are presented in Sec. IV, while the Appendix reports in detail the derivation of the set of equations.

II. PLASMA-FEL MODEL AND EQUATIONS

The plasma-FEL system [8–11] consists of an intense electron beam, which is injected inside the wiggler cavity loaded with a plasma. A Langmuir wave, unstable in the presence of the beam, develops inside the plasma and couples with the transverse field excited in the wiggler by the usual FEL instability. The simultaneous occurrence of these two instabilities constitutes the principal feature of this system.

The Langmuir electrostatic wave and the transverse wave couple resonantly if their frequencies (respectively ω_L and ω_T) and wave numbers (k_L and k_T) satisfy the usual matching conditions

$$\omega_L=\omega_T, \quad k_L=k_T+k_W , \tag{1}$$

where k_W is the wave number of the magnetic field of the wiggler. The pseudowave associated with the rippled magnetostatic ($\omega_W=0$) field of the wiggler is the third partner in this parametric decay process. It is important to note that if we call $\theta_T=(k_T+k_W)z-\omega_T t$ and $\theta_L=k_L z-\omega_L t$ the phases of the electrons of the beam in the transverse and longitudinal fields, respectively, the above matching conditions (1) can be derived by imposing that $\theta_T=\theta_L$ for all z and t, so that a unique phase θ is present in the interaction process. We assume that the electron density n_b of the beam is much smaller than the density n_p of the ambient plasma and that the frequencies of both transverse and Langmuir waves are close to ω_p, the electron plasma frequency.

In the limit of small wiggler parameter $a_{W0}\ll 1$, the dispersion relation of the Langmuir waves is $\omega_L=(3k_L^2 v_{\rm th}^2+\omega_p^2)^{1/2}=[3v_{\rm th}^2(k_W+k_T)^2+\omega_p^2]^{1/2}$, where $v_{\rm th}=(\kappa T_e/m)^{1/2}\ll c$ is the thermal velocity of the electrons of the plasma (κ is the Boltzmann constant), while that of the transverse waves is given by $\omega_T=(c^2k_T^2+\omega_p^2)^{1/2}$. The two frequencies ω_T and ω_L as functions of k, as well as the matching conditions (1), are represented in Fig. 1. It can be seen that the waves excited have frequencies $\omega_T=\omega_L=[\omega_p^2+3c^2v_{\rm th}^2k_W^2/(c-\sqrt{3}v_{\rm th})^2]^{1/2}\approx\omega_p$ and wave numbers $k_T=\sqrt{3}v_{\rm th}k_W/(c-\sqrt{3}v_{\rm th})\approx\sqrt{3}(v_{\rm th}/c)k_W$ for the transverse wave and $k_L=ck_W/(c-\sqrt{3}v_{\rm th})\approx k_W$ for the longitudinal wave, respectively. It also follows that it is possible to excite Langmuir waves with phase velocities $\omega_L/(k_T+k_W)\approx\omega_p/k_W$ larger than the velocity of light c when the condition $\omega_p>ck_W$ is satisfied.

As we have already said, the present scheme differs from that described by Bobin [7] in the following point: we propose to excite a Langmuir wave with frequency $\omega_L\approx\omega_p$, while in Bobin's scheme the excited electrostatic wave has a much larger frequency, i.e., $\omega_L\approx k_Lv_b$ (v_b being the velocity of the injected electron beam) and a phase velocity that is approximately equal to that of the beam. As a consequence, the electric field associated with this wave is primarily that produced by the perturbation of the density of the beam, while the plasma plays only a secondary role in the process [12].

As described in more details in the Appendix, we assume an elicoidal wiggler and give the radiation field in terms of its vector potential **A**, while the Langmuir wave is given in terms of the associated density modulation $\delta n/n_p$, where n_p is the unperturbed plasma density. As shown in the Appendix, the dynamics of the physical system is described by the following set of self-consistent nonlinear equations:

$$\frac{d}{d\tau}\theta_j=\frac{p_j}{\sqrt{p_j^2+1+a_{W0}^2}}-\beta_R\ ,$$

$$\frac{d}{d\tau}p_j=-i\frac{\Omega_b^2}{2}\{\langle b\rangle e^{i\theta_j}-\text{c.c.}\}-a_{W0}\left\{A_T\frac{e^{i\theta_j}}{\sqrt{p_j^2+1+a_{W0}^2}}+\text{c.c.}\right\}-\Omega_p^2\{A_Le^{i\theta_j}+\text{c.c.}\}\ , \qquad (2)$$

$$\frac{d}{d\tau}A_T=S_1\frac{1}{N}\sum_j\frac{e^{-i\theta_j}}{\sqrt{p_j^2+1+a_{W0}^2}}-iS_2A_L\ ,$$

$$\frac{d}{d\tau}A_L=S_3\langle b\rangle-iS_4A_T\ ,$$

where the index j runs from 1 to N, the number of electrons of the beam in the wavelength $\lambda_L=2\pi/k_L$. $\theta_j=(k_T+k_W)z_j-\omega_Tt=k_Lz_j-\omega_Lt$ is the phase of the electrons in the fields of the two waves, while $p_j=\beta_j\gamma_j$, where $\beta_j=v_j/c$, v_j being the axial velocity and

$$\gamma_j=(1-\beta_j^2-\beta_{\perp j}^2)^{-1/2}=[(1+a_{W0}^2)/(1-\beta_j^2)]^{1/2}=(p_j^2+1+a_{W0}^2)^{1/2}$$

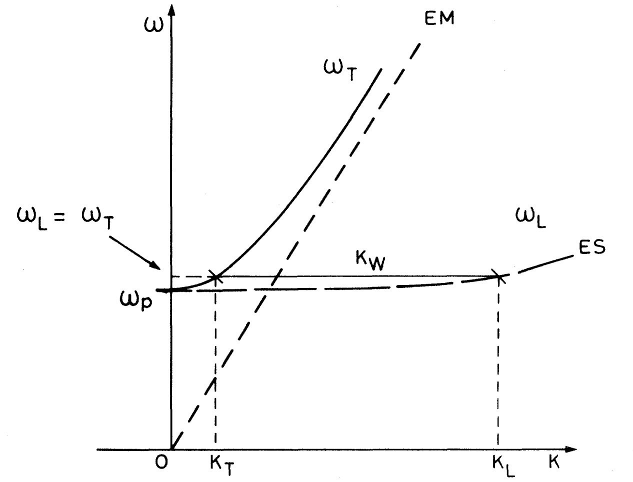

FIG. 1. Schematic representation of the dispersion relation ω vs k of the plasma modes $\omega_T(k)$ and $\omega_L(k)$.

the Lorentz factor of the electrons of the beam. Furthermore, the scaled time τ is defined as $\tau=c(k_T+k_W)t$. $S_1=\Omega_b^2a_{W0}/(4\beta_R)$, $S_2=\Omega_p^2a_{W0}/(4\beta_R)$, $S_3=\Omega_b^2/(2\beta_R)$, and $S_4=a_{W0}/(2\beta_R)$ are the coefficients of the equations, with $\Omega_b=\omega_b/[c(k_T+k_W)]$, $\Omega_p=\omega_p/[c(k_T+k_W)]$, $\beta_R=\omega_T/[c(k_T+k_W)]$, and $\omega_b^2=4\pi n_be^2/m$ the plasma frequency of the beam. Finally, $\langle b\rangle=(1/N)\sum_{j=1}^Ne^{-i\theta_j}$ is the bunching factor of the electrons of the beam.

As shown in the Appendix, these equations can be deduced from the Maxwell equations, the fluid equations describing a nearly cold plasma with fixed positive ions, and the fully relativistic equations of motion of the electrons of the beam. They give the slow time evolution of the field amplitudes and of the dynamical variables of the electrons of the beam in the framework of the slowly varying amplitude approximation. It is important to stress the fact that we do not impose any restrictions on either the values or the variations in time of the Lorentz factors γ_j during the whole process. This allows us to treat the dynamics of the beam also outside the usual Compton limit [13].

One can show that Eqs. (2) admit the constant of motion

$$\langle p\rangle+\frac{4\beta_R}{\Omega_b^2}|A_T|^2+\frac{2\beta_R\Omega_p^2}{\Omega_b^2}|A_L|^2=\text{const}\ , \qquad (3)$$

where $\langle p\rangle=(1/N)\sum_{j=1}^Np_j$ is the average momentum of the beam. It is easy to see that each physical element appearing in (3), i.e., the electron beam and the transverse and longitudinal waves, can play the role of source and therefore feed energy into the other ones. In this sense, the plasma FEL is conventionally used to generate or amplify electromagnetic waves by transferring the kinetic energy of the beam to the transverse field [8–11]. Conversely, the average momentum of the beam can be increased at the expense of the energy stored in the fields (the inverse plasma FEL).

III. NUMERICAL RESULTS AND PHYSICAL INTERPRETATION

The system of equations (2) has been integrated numerically by assuming that at the initial time $\tau=0$ the beam is strictly monokinetic and the bunching factor $\langle b\rangle$ is almost zero. As regards the initial value of the fields, different choices can be made, corresponding to different experimental scenarios. In fact, if at the initial time a given value $A_T(0)$ is assigned to the transverse field, while $A_L(0)\approx 0$, this corresponds to assuming that a transverse wave is injected inside the system. On the other hand, the complementary choice $A_T(0)\approx 0$ and $A_L(0)\neq 0$ corresponds to the case of direct excitation of the Langmuir wave inside the plasma, as done, for instance, in the laser beat acceleration experiments.

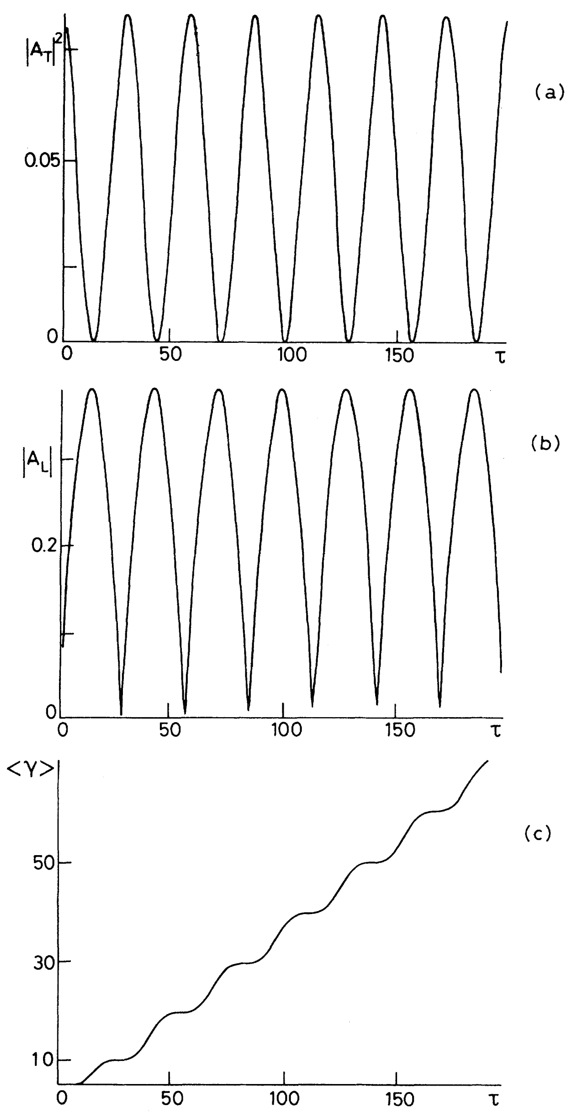

FIG. 2. (a) $|A_T|^2$ vs τ, (b) $|A_L|$ vs τ, and (c) $\langle\gamma\rangle$ vs τ, for $\Omega_p^2=1.2354$, $a_{W0}=0.316$, $\Omega_b=10^{-3}$, $\gamma(0)=5$, $\mu=10^{-3}$, $|A_T(0)|^2=0.09$, and $|A_L(0)|=0$.

A typical numerical output is shown in Fig. 2, where $|A_T|^2$ [Fig. 2(a)] and $|A_L|$ [Fig. 2(b)] are presented versus τ. Figure 2(c) gives the time evolution of the average Lorentz factor $\langle\gamma\rangle$. This case was run with $\Omega_p^2=1.2354$, $a_{W0}=0.316$, $\Omega_b=10^{-3}$, $\gamma(0)=5$, $\mu=10^{-3}$, $|A_T(0)|^2=0.5$, and $|A_L(0)|=0$.

It can be seen that the transverse and longitudinal fields exchange their energy with a definite period in time [see Figs. 2(a) and 2(b)] and that the electrons of the beam gain energy at each period under the action of the electrostatic field and do so in a cumulative way [Fig. 2(c)]. After several amplitude oscillations the average value of γ reaches the saturation (see Fig. 3).

By collecting the data in a graph showing the saturation value of $\langle\gamma\rangle$ as a function of Ω_p for $a_{W0}=0.316$, a typical resonance structure appears (see Fig. 4). In particular, it is worth noting that two different resonances develop, corresponding to two values of Ω_p, namely, $\Omega_p\approx 0.887$ and 1.112. While the first of these resonances takes place for values $\Omega_p<1$, corresponding therefore to excited Langmuir waves with phase velocity less than c, the second resonance occurs for $\Omega_p>1$ and corresponds to waves with phase velocity larger than c. The value of $\langle\gamma\rangle$ at the peak of the resonances is larger in the case $\Omega_p>1$, this being due to the fact that the electric field of the Langmuir waves scales proportionally to the phase velocity of the wave.

The existence of these resonances and their physical meaning can also be investigated by the following simple analysis of Eqs. (2). If we start at a time τ_0 when the acceleration process is under way and the bunching factor $\langle b\rangle=(1/N)\sum_{j=1}^{N}e^{-i\theta_j}$ is already appreciably different from zero, $\beta_j\approx 1$ and hence $\gamma_j\gg 1$ for practically all j, the equations can be simplified by neglecting terms $O(1/\gamma_j)$, in the equations for p_j and A_T and assuming the form

$$\frac{d}{d\tau}\theta_j=1-\beta_R\ ,$$

$$\frac{d}{d\tau}p_j=-i\frac{\Omega_b^2}{2}\{\langle b\rangle e^{i\theta_j}-\text{c.c.}\}-\Omega_p^2\{A_L e^{i\theta_j}+\text{c.c.}\}\ ,$$

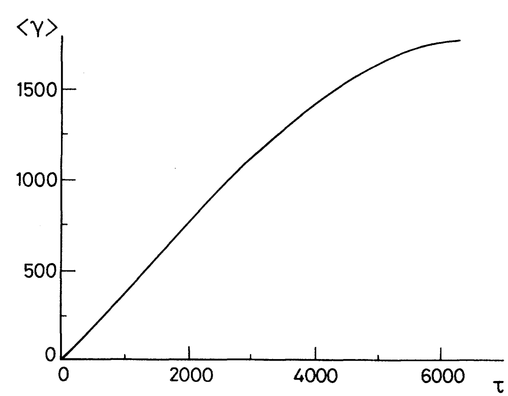

FIG. 3. $\langle\gamma\rangle$ vs τ for the same parameters as in Fig. 2, but over a larger time scale.

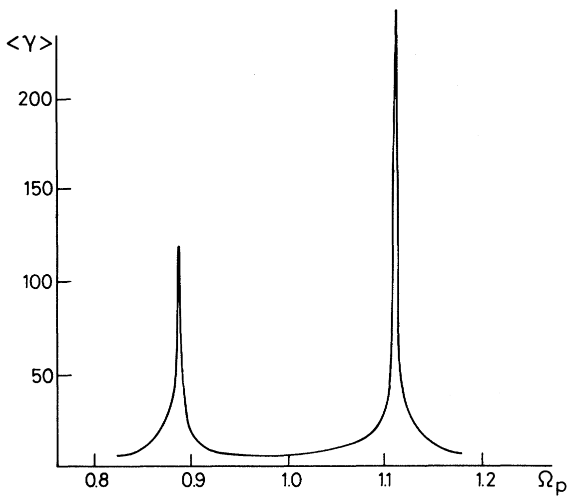

FIG. 4. Saturation value of $\langle\gamma\rangle$ vs Ω_p, for $\Omega_b=2.36\times10^{-2}$, $a_{W0}=0.316$, $|A_T(0)|^2=0$, and $|A_L(0)|=0.3$.

$$\frac{d}{d\tau}A_T=-iS_2A_L\ ,$$

$$\frac{d}{d\tau}A_L=S_3\langle b\rangle-iS_4A_T\ .$$

The equations for the electron phases θ_j can then be integrated, yielding $\theta_j\approx(1-\beta_R)\tau+\theta_{j0}$, where θ_{j0} are the values of the phases at $\tau=\tau_0$. The equation for A_L gives

$$A_L=C_1e^{i\sqrt{S_2S_4}\tau}+C_2e^{-i\sqrt{S_2S_4}\tau}-\frac{iS_3(1-\beta_R)\langle b\rangle_0e^{-i(1-\beta_R)t}}{S_2S_4-(1-\beta_R)^2}\ ,$$

where C_1 and C_2 are two suitable complex constants of integration depending on the previous history of the system. By inserting A_L into the equation for the momentum p_j, averaging over all electrons, and considering that $S_2S_4=\Omega_p^2a_{W0}^2/(8\beta_R^2)$, one finally gets

$$\frac{d\langle p\rangle}{d\tau}=-\Omega_p^2(C_1e^{i(\sqrt{S_2S_4}+1-\beta_R)\tau}+C_2e^{i(-\sqrt{S_2S_4}+1-\beta_R)\tau}+\text{c.c.})\ .$$

This relation reveals the existence of two resonances that correspond to the two conditions $1-\beta_R+a_{W0}\Omega_p/(2\sqrt{2}\beta_R)=0$ and $1-\beta_R-a_{W0}\Omega_p/(2\sqrt{2}\beta_R)=0$, respectively. If we take into account that for a nearly cold plasma $\beta_R\approx\Omega_p$, the preceding two conditions can be cast in the simpler form $\Omega_p=1\pm a_{W0}/(2\sqrt{2})$. If we define the Larmor frequency of the electrons of the beam in the plasma loaded wiggler as $\omega_{ce}=e(B_W+B_D)/(mc)$, where B_D is the diamagnetic magnetostatic field produced by the equilibrium plasma, and $T_{ce}=2\pi/\omega_{ce}$ as the Larmor period, the above resonance conditions require that the distance cT_{ce} traveled by the electrons of the beam in the cyclotron period equals the distance traveled in the same time by the electrostatic wave $(\omega_p/k_W)T_{ce}$, plus or minus half of a Langmuir wavelength $\lambda_W/2$.

When the resonance condition is satisfied, the acceleration of the electrons is considerable. For instance, in the case of Fig. 3, with $a_{W0}=0.316$ and $\Omega_p=1+a_{W0}/(2\sqrt{2})\approx1.112$, the maximum value of $\langle\gamma\rangle$ is about 1800 and the acceleration gradient is about 130 MeV/m with a wiggler wavelength λ_W of 1 cm.

Figure 5 gives the normalized distribution of energy of the electrons of the beam versus γ at $\tau=3000$ and 6900,

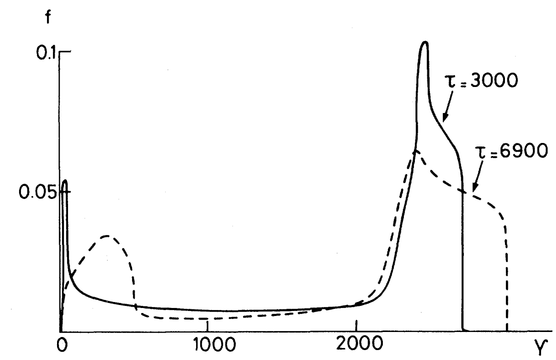

FIG. 5. Normalized distribution function $f(\gamma)$ $[\int f(\gamma)d\gamma=1]$ vs γ for the case of Fig. 3, at $\tau=3000$ (solid line) and $\tau=6900$ (dashed line).

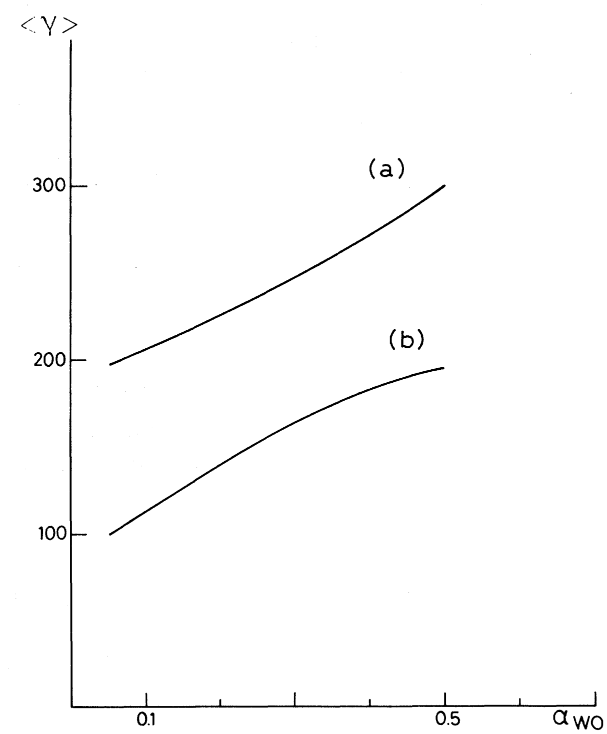

FIG. 6. Saturation value of $\langle\gamma\rangle$ vs a_{W0} in the condition of resonance for (a) $|A_T(0)|^2=0$ and $|A_L(0)|=0.3$ and (b) $|A_T(0)|^2=0.04$ and $|A_L(0)|=0$.

for the same parameters as in Fig. 3. It can be seen that at $\tau=6900$, a large fraction, about 25%, of the electrons of the beam acquires energies greater than 1.3 GeV, with an acceleration gradient of about 0.2 GeV/m.

Figure 6 gives the saturation value of $\langle \gamma \rangle$ at resonance versus a_{W0}, in the case (a) of applied electrostatic field $|A_L(0)|=0.3$ and $|A_T(0)|^2=0$ and in the case (b) of applied transverse field $|A_T(0)|^2=0.04$ and $|A_L(0)|=0$. It shows that there is an advantage in increasing the characteristic parameter a_{W0} of the wiggler. For $a_{W0}\approx 1$, however, the external magnetic field of the wiggler modifies considerably the dispersion of the plasma and the present analysis loses its validity.

IV. CONCLUSIONS

We have demonstrated that a plasma-FEL system can be conveniently used as a compact high-gradient accelerator. The strong plasma wave that accelerates the electrons of the beam can be excited by simply injecting into the wiggler cavity a transverse wave with frequency close to the plasma frequency ω_p. If we work with not very large plasma densities, e.g., with n_p of the order of 10^{13} cm^{-3}, we can use transverse waves with frequencies in the microwave region ($\omega\approx 10$ GHz).

We have also shown that there exist two resonance conditions that are represented by the relation $\beta_R=1\pm a_{W0}/(2\sqrt{2})$. When these relations are satisfied, the electrons of the beam acquire energy from the electrostatic field of the Langmuir waves in a cumulative way. Although the process eventually saturates, the acceleration reached in conditions of resonance is considerable, values of energy of about 0.1 GeV and more being easily reached at saturation. For instance, with $n_p\approx 10^{13}$ cm^{-3}, a wiggler wavelength of about 1 cm, and a wiggler magnetic field of 0.3 T, we are able to obtain an acceleration gradient of about 150 MeV/m. Moreover, from the analysis of the energy distribution of the electrons of the beam, we see that a large fraction, about 25% or more, of the electrons possess an energy that is twice the average value and this fraction can certainly be increased by preparing the beam in a suitable way at the initial time of the process.

APPENDIX: EQUATIONS OF THE MODEL

We start with the usual wave equation for the (transverse) vector potential $\mathbf{A}(z,t)$ in one dimension, namely,

$$\left[\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial z^2}\right]\mathbf{A}(z,t)=4\pi c\mathbf{J}_\perp , \tag{A1}$$

where $\mathbf{J}_\perp$ is the transverse component of the total current density $\mathbf{J}=\mathbf{J}_b+\mathbf{J}_p$ and

$$\mathbf{J}_p=-en\mathbf{u} , \tag{A2}$$

$$\mathbf{J}_b=-en_\perp\sum_s \mathbf{v}_s(t)\delta(z-z_s(t)) . \tag{A3}$$

Equation (A2), with n and $\mathbf{u}$ the plasma density and average speed, respectively, gives the usual plasma current density, while Eq. (A3) gives the beam current density. The electron beam is modeled here as an ensemble of uniformly charged sheets moving along the z axis, with $-en_\perp$ the charge per unit surface of each sheet and $v_s(t)$ the common velocity of all electrons that lie on the same sheet.

To Eq. (A1) one must add the Poisson equation for the axial component E_z of the electric field

$$\frac{\partial E_z}{\partial z}=4\pi\rho , \tag{A4}$$

where $\rho=\rho_b+\rho_p$ and

$$\rho_p=en_p-en , \tag{A5}$$

$$\rho_b=-en_\perp\sum_s \delta(z-z_s(t)) . \tag{A6}$$

In these equations, n_p is the number per unit volume of the fixed neutralizing ions of the plasma, while ρ_b is the beam charge density.

The plasma is described as a nearly cold electron fluid by the two equations

$$\frac{\partial}{\partial t}n+\frac{\partial}{\partial z}(nu_z)=0 , \tag{A7}$$

$$\frac{\partial}{\partial t}u_z+u_z\frac{\partial}{\partial z}u_z=-\frac{e}{m}E_z-\frac{\theta^2}{n}\frac{\partial}{\partial z}n -\frac{e^2}{2m^2c^2}\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{A}_W)^2 . \tag{A8}$$

In (A8), $\mathbf{A}_W=(a_W/\sqrt{2})(\bar{\mathbf{e}}e^{-ik_Wz}+\text{c.c.})$ is the vector potential of the helical magnetostatic field of the wiggler, with $\bar{\mathbf{e}}=(1/\sqrt{2})(\mathbf{e}_x+i\mathbf{e}_y)$ and $\theta^2=3v_{\text{th}}^2$, where $v_{\text{th}}=\sqrt{T_e/m}$ is the thermal velocity of the electrons of the plasma. As usual, $\mathbf{u}_\perp$, the transverse component of the plasma speed, is written in terms of the total vector potential as

$$\mathbf{u}_\perp=\frac{e}{mc}(\mathbf{A}+\mathbf{A}_W) . \tag{A9}$$

Equations (A1), (A4), (A7), and (A8) constitute a closed set for the description of the physical system once one adds the (fully relativistic) equations of motion of each single sheet in the beam, namely,

$$\frac{d}{dt}z_s(t)=v_s(t) , \tag{A10}$$

$$\frac{d}{dt}p_s(t)=-\left\{\frac{e}{mc}E_z(z,t)+\frac{e^2}{2m^2c^3\gamma_s(t)}\times\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{A}_W)^2\right\}_{z=z_s(t)}, \tag{A11}$$

where $v_s(t)=c\beta_s(t)$ is the axial velocity, $p_s=\beta_s\gamma_s$, the γ_s are the Lorentz factors of the electrons of the beam, and

$$\mathbf{v}_{s\perp}(t)=\frac{e}{mc\gamma_s(t)}[\mathbf{A}+\mathbf{A}_W]_{z=z_s(t)} . \tag{A12}$$

We assume that the plasma is in equilibrium with the static field of the wiggler before the injection of the beam, with a constant density $n = n_p$, no average motion in the axial direction $u_z = 0$, and no axial field $E_z = 0$, but with a transverse diamagnetic vector potential $\mathbf{A}_d = -\omega_p^2/(\omega_p^2 + c^2 k_W^2)\mathbf{A}_W$, where $\omega_p^2 = 4\pi n_p e^2/m$.

The beam is then supposed to be so weak as to produce only small deviations of all relevant parameters from the equilibrium values. Disregarding nonlinear (bilinear) terms and writing the displacement of the vector potential in the wiggler reference system as $\delta\mathbf{A} = \delta A\,\bar{\mathbf{e}} + \text{c.c.}$, it is rather straightforward to write the two equations for the normalized displacements $\delta A_T = e/mc^2\delta A$ and $\delta n/n_p$, respectively,

$$\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2} + \omega_p^2\right]\delta A_T = -\frac{4\pi e^2 n_\perp a_{W0}}{\sqrt{2}m}\sum_s \frac{1}{\gamma_s(t)} e^{-ik_W z}\delta(z - z_s(t)) - \omega_p^2\frac{a_{W0}}{\sqrt{2}} e^{-ik_W z}\frac{\delta n}{n_p}\,, \tag{A13}$$

$$\left[\frac{\partial^2}{\partial t^2} - \theta^2\frac{\partial^2}{\partial z^2} + \omega_p^2\right]\frac{\delta n}{n_p} = -\frac{4\pi e^2 n_\perp}{m}\sum_s \delta(z - z_s(t)) + c^2\frac{a_{W0}}{\sqrt{2}}\frac{\partial^2}{\partial z^2}(e^{ik_W z}\delta A_T + \text{c.c.})\,, \tag{A14}$$

where

$$a_{W0} = \frac{e a_W}{mc^2}\frac{1}{1 + \omega_p^2/c^2 k_W^2}\,. \tag{A15}$$

In the analysis that follows we shall suppose that the right-hand sides of Eqs. (A13) and (A14) are to be considered small quantities due to the assumed smallness of the wiggler parameter a_{W0} and of the ratio n_b/n_p between the beam density and the plasma density. It is convenient, at this point, to apply a Fourier transform in space to Eqs. (A13) and (A14) and discuss the time evolution of the spectrum of the radiation field $\overline{\delta A_T}$ and that of the longitudinal field $\overline{\delta n}/n_p$.

We can simplify the treatment even further by disregarding all waves that propagate in the negative z direction and writing the two first-order equations for the progressive waves only,

$$\left[\frac{\partial}{\partial t} + i\omega_T(k)\right]\overline{\delta A_T} = -i\epsilon\frac{4\pi e^2 n_\perp a_{W0}}{2\sqrt{2}m\,\omega_T(k)}\sum_s \frac{1}{\gamma_s(t)} e^{-i(k_W + k)z_s(t)} - i\epsilon\frac{\omega_p^2 a_{W0}}{2\sqrt{2}\omega_T(k)}\frac{\overline{\delta n}}{n_p}(k + k_W) - \epsilon\frac{\partial}{\partial t_1}\overline{\delta A_T}, \tag{A16}$$

$$\left[\frac{\partial}{\partial t} + i\omega_L(k)\right]\frac{\overline{\delta n}}{n_p} = -i\epsilon\frac{4\pi e^2 n_\perp}{2m\,\omega_L(k)}\sum_s e^{-ikz_s(t)} - i\epsilon\frac{c^2k^2 a_{W0}}{2\sqrt{2}\omega_L(k)}[\overline{\delta A_T}(k - k_W) - \overline{\delta A}_T^*(-k - k_W)] - \epsilon\frac{\partial}{\partial t_1}\frac{\overline{\delta n}}{n_p}\,,$$

where $\omega_T(k) = \sqrt{c^2k^2 + \omega_p^2}$ and $\omega_L(k) = \sqrt{\theta^2k^2 + \omega_p^2}$. A formal parameter of smallness ϵ appears in these two equations, in which we have also introduced the slow time scale $t_1 = \epsilon t$, as is customary in the multiple time scale procedure. At zeroth order in the perturbation treatment

$$\overline{\delta A}_t^{(0)} = a_T(k, \epsilon t)e^{-i\omega_T(k)t}\,, \qquad \frac{\overline{\delta n}^{(0)}}{n_p} = a_L(k, \epsilon t)e^{-i\omega_L(k)t}\,, \tag{A17}$$

and we may also eliminate all secular behaviors in the first-order quantities by imposing that the slow time behaviors of the two spectra $a_T(k)$ and $a_L(k)$ are given by

$$\frac{\partial}{\partial t_1}a_T = -i\frac{4\pi e^2 n_\perp a_{W0}}{2\sqrt{2}m\,\omega_T(k)}\sum_s \frac{1}{\gamma_s(t)} e^{-i\theta_s(t)} - i\frac{\omega_p^2 a_{W0}}{2\sqrt{2}\omega_T(k)}a_L(k + k_W)e^{-i[\omega_L(k + k_W) - \omega_T(k)]t}\,,$$
$$\frac{\partial}{\partial t_1}a_L = -i\frac{4\pi e^2 n_\perp}{2m\,\omega_L(k)}\sum_s e^{-i[kz_s(t) - \omega_L(k)t]} - i\frac{c^2k^2 a_{W0}}{2\sqrt{2}\omega_L(k)}[a_T(k - k_W)e^{-i[\omega_T(k - k_W) - \omega_L(k)]t} + a_T^*(-k - k_W)e^{-i[\omega_T(k + k_W) - \omega_L(k)]t}]\,, \tag{A18}$$

where the angle $\theta_s(t)$ is given by $\theta_s(t) = (k + k_W)z_s(t) - \omega_T(k)t$. By applying the same procedure to Eqs. (A10)–(A12), which give the dynamics of the beam, we may write the slow time evolution of the momentum p_s of each single electron as

$$\frac{\partial}{\partial t_1}p_s=\frac{2\pi e^2 n_\perp}{mc}\sum_r \operatorname{sgn}(z_s(t)-z_r(t))$$
$$-i\frac{ca_{W0}}{\sqrt{2}\gamma_s}\left[\frac{1}{2\pi}\int dk(k+k_W)e^{i\theta_s}a_T(k)-\text{c.c.}\right]$$
$$-i\frac{\omega_p^2}{2\pi c}\int\frac{dk}{k}e^{i[kz_s(t)-\omega_L(k)t]}a_L(k)\ . \qquad \text{(A19)}$$

Equations (A18) and (A19) define the time behavior of the Fourier spectra of both transverse and longitudinal signals.

We shall now assume that the two spectra $a_T(k)$ and $a_L(k)$ are very narrow and centered around some definite values of k, say, k_T and k_L, respectively, in such a way that, to dominant order,

$$\delta A_T=M_T(z,t)e^{i[k_T z-\omega_T(k_T)t]}\ ,$$
$$\frac{\delta n}{n_p}=M_L(z,t)e^{i[k_L z-\omega_L(k_L)t]}+\text{c.c.} \qquad \text{(A20)}$$

If we introduce the preceding hypothesis of narrow spectra into the basic equations (A18) and (A19), it is possible to rewrite them in a form that gives directly the time behavior of the two amplitudes M_T and M_L. These equations in turn take a particularly simple form if we make the additional hypotheses that

$$k_L=k_T+k_W,\quad \omega_L(k_L)=\omega_T(k_T)\ . \qquad \text{(A21)}$$

At last, averaging over the smaller wavelength of the problem, we get the equations in the final form

$$\left[\frac{\partial}{\partial t_1}+\omega_r'(k_T)\frac{\partial}{\partial z_1}\right]M_T$$
$$=-i\frac{\omega_b^2 a_{W0}}{2\sqrt{2}\omega_T(k_T)}\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\gamma_j(t)}e^{-i\theta_j(t)}$$
$$-i\frac{\omega_p^2 a_{W0}}{2\sqrt{2}\omega_T(k_L)}M_L\ ,$$
$$\left[\frac{\partial}{\partial t_1}+\omega_L'(k_L)\frac{\partial}{\partial z_1}\right]M_L$$
$$=-i\frac{\omega_b^2}{2\omega_L(k_L)}\frac{1}{N}\sum_{j=1}^{N}e^{-i\theta_j(t)}-i\frac{a_{W0}c^2k_L^2}{2\sqrt{2}\omega_L(k_L)}M_T\ , \qquad \text{(A22)}$$
$$\frac{\partial}{\partial t_1}p_j=-i\frac{\omega_b^2}{2ck_L}\left[\left[\frac{1}{N}\sum_{s=1}^{N}e^{-i\theta_s(t)}\right]e^{i\theta_j(t)}-\text{c.c.}\right]$$
$$-i\frac{a_{W0}}{\sqrt{2}}ck_L\left[\frac{e^{i\theta_j(t)}}{\gamma_j(t)}M_T-\text{c.c.}\right]$$
$$-i\frac{\omega_p^2}{ck_L}(e^{i\theta_j(t)}M_L-\text{c.c.})\ ,$$

where the index j now runs over all electrons in a length equal to the longitudinal wavelength $\lambda_L=2\pi/k_L$. Furthermore, in (A22), $z_1=\epsilon z$, $\omega_b^2=4\pi n_b e^2/m$, $\theta_j(t)=(k+k_W)z_j(t)-\omega_T(k_T)t$, and ω_T' and ω_L' are the group velocities of the transverse and longitudinal wave packets, respectively. Equations (2) in the text are obtained from Eqs. (A22) by assuming that the amplitudes M_T and M_L do not depend on z and change finally to $A_T=i(M_T/\sqrt{2})$ and $A_L=iM_L$.

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